

# Quasi-classical generalized CRF structures

by

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**ABSTRACT.** In an earlier paper, we studied manifolds  $M$  endowed with a generalized F structure  $\Phi \in \text{End}(TM \oplus T^*M)$ , skew-symmetric with respect to the pairing metric, such that  $\Phi^3 + \Phi = 0$ . Furthermore, if  $\Phi$  is integrable (in some well-defined sense),  $\Phi$  is a generalized CRF structure. In the present paper we study quasi-classical generalized F and CRF structures, which may be seen as a generalization of the holomorphic Poisson structures (it is well known that the latter may also be defined via generalized geometry). The structures that we study are equivalent to a pair of tensor fields  $(A \in \text{End}(TM), \pi \in \wedge^2 TM)$  where  $A^3 + A = 0$  and some relations between  $A$  and  $\pi$  hold. We establish the integrability conditions in terms of  $(A, \pi)$ . They include the facts that  $A$  is a classical CRF structure,  $\pi$  is a Poisson bivector field and  $\text{im } A$  is a (non)holonomic Poisson submanifold of  $(M, \pi)$ . We discuss the case where either  $\ker A$  or  $\text{im } A$  is tangent to a foliation and, in particular, the case of almost contact manifolds. Finally, we show that the dual bundle of  $\text{im } A$  inherits a Lie algebroid structure and we briefly discuss the Poisson cohomology of  $\pi$ , including an associated spectral sequence and a Dolbeault type grading.

## 1 Introduction

In this paper, the differentiable manifolds and the differential geometric objects are  $C^\infty$ -smooth and the notation is that used in most textbooks on differential geometry.

Generalized geometry [9] is the geometry of structures defined on the big tangent bundle  $\mathbf{T}M = TM \oplus T^*M$  of a differentiable manifold  $M^m$ , endowed with the pairing metric

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)) \quad (1.1)$$

and the Courant bracket

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))). \quad (1.2)$$

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In generalized geometry the most popular subject is generalized complex structures [8]. A generalized almost complex structure is a complex, maximal  $g$ -isotropic subbundle  $L \subseteq \mathbf{T}^c M = T^c M \oplus T^{*c} M$  such that  $L \cap \bar{L} = 0^1$ . Equivalently, the structure is defined by the endomorphism  $\mathcal{J}$  of  $\mathbf{T}M$  with  $\pm i$ -eigenbundles  $L, \bar{L}$ , which is characterized by the properties (i)  $\mathcal{J}$  is  $g$ -skew-symmetric, (ii)  $\mathcal{J}^2 = -Id$ . Furthermore, the structure is integrable, or generalized complex, if  $L$  is closed under the Courant bracket.

The endomorphism  $\mathcal{J}$  may be expressed by a matrix of classical tensors [8, 15], which we call the *tensor components* of  $\mathcal{J}$ ,

$$\mathcal{J} \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \sharp_\pi \\ \flat_\sigma & -A^* \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix} \quad (1.3)$$

where  $X \in TM, \alpha \in T^*M, A \in \text{End}(TM), \sigma \in \Omega^2(M)$  ( $\Omega^k(M)$  is the space of differential  $k$ -forms of  $M$ ),  $\pi \in \chi^2(M)$  ( $\chi^k(M)$  is the space of  $k$ -vector fields, i.e., contravariant, skew-symmetric tensor fields of  $M$ ),  $\flat_\sigma X = i(X)\sigma$ ,  $\sharp_\pi \alpha = i(\alpha)\pi$  and  $*$  denotes transposition ( $A^* \alpha = \alpha \circ A$ ). Then,  $g$ -skew-symmetry holds and  $\mathcal{J}^2 = -Id$  is equivalent to

$$A^2 + \sharp_\pi \circ \flat_\sigma = -Id, \sharp_\pi \circ A^* = A \circ \sharp_\pi, \flat_\sigma \circ A = A^* \circ \flat_\sigma. \quad (1.4)$$

The integrability of the structure is equivalent to the set of conditions [3, 15]

$$\begin{aligned} [\pi, \pi] &= 0, \quad R_{(\pi, A)} = 0, \quad \mathcal{N}_A(X, Y) = \sharp_\pi[i(X \wedge Y)d\sigma], \\ d\sigma_A(X, Y, Z) &= \sum_{Cycl(X, Y, Z)} d\sigma(AX, Y, Z), \end{aligned} \quad (1.5)$$

where  $[\cdot, \cdot]$  is the Schouten-Nijenhuis bracket [13],

$$\mathcal{N}_A(X, Y) = [AX, AY] - A[X, AY] - A[AX, Y] + A^2[X, Y] \quad (1.6)$$

is the Nijenhuis tensor,

$$R_{(\pi, A)}(X, \alpha) = \sharp_\pi[L_X(A^* \alpha) - L_{AX} \alpha] - (L_{\sharp_\pi \alpha} A)(X) \quad (1.7)$$

is the Schouten concomitant ( $L$  denotes Lie derivative) and  $\sigma_A(X, Y) = \sigma(AX, Y)$ .

If  $\mathcal{J}(TM) \subseteq TM$  and  $\mathcal{J}(T^*M) \subseteq T^*M$ , equivalently,  $\sigma = 0, \pi = 0$ ,  $\mathcal{J}$  reduces to a classical almost complex structure  $A$  and the integrability conditions (1.5) reduce to the integrability condition  $\mathcal{N}_A = 0$ .

We propose to call  $\mathcal{J}$  a *quasi-classical* structure if  $\mathcal{J}(TM) \subseteq TM$ , equivalently,  $\sigma = 0$ . It is known that in this case the integrability conditions (1.5) are equivalent to the fact that  $A$  is a complex structure and  $\pi$  is a holomorphic Poisson structure on  $(M, A)$ , e.g., [7]. Indeed, the third condition (1.5) becomes  $\mathcal{N}_A = 0$ , i.e.,  $A$  is complex and  $M$  has the local, complex analytic coordinates  $(z^i)$ . Then, since  $R$  is a tensor, it suffices to check  $R = 0$  for  $X = \partial/\partial z^i, \alpha = dz^j$  and  $X = \partial/\partial \bar{z}^i, \alpha = d\bar{z}^j$ . In the first case,  $R = 0$  becomes

$$A[\sharp_\pi(dz^j), \frac{\partial}{\partial \bar{z}^i}] = i[\sharp_\pi(dz^j), \frac{\partial}{\partial \bar{z}^i}],$$

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<sup>1</sup>The upper index  $c$  denotes complexification and the bar denotes complex conjugation.

which holds iff  $\pi$  is a holomorphic Poisson structure, because  $\pi$  has no component of complex type  $(1, 1)$  and  $\sharp_\pi(dz^j)$  is an  $i$ -eigenvector of  $A$ . In the second case,  $R = 0$  is implied by  $\pi^{ij} = \pi^{ij}(z^k)$ , which, in turn, is implied by  $[\pi, \pi] = 0$ .

In [17] we studied a more general type of generalized structure that corresponds to K. Yano's F structure [18]. An F structure is an endomorphism  $A \in \text{End}(TM)$  such that  $A^3 + A = 0$ . Then,  $A$  has the eigenvalues  $\pm i, 0$  with the corresponding eigenbundles  $H, \bar{H} \subseteq T^cM$ ,  $Q = \ker A \subseteq TM$  and with  $P = \text{im } A \subseteq TM$  such that  $P^c = H \oplus \bar{H}$  and  $TM = P \oplus Q$ . The projections that correspond to the decomposition  $T^cM = H \oplus \bar{H} \oplus Q^c$  are given by

$$pr_H = -\frac{1}{2}(A^2 + iA), pr_{\bar{H}} = -\frac{1}{2}(A^2 - iA), pr_Q = A^2 + Id, pr_P = -A^2. \quad (1.8)$$

Furthermore,  $H$  is an almost CR structure and, if it is closed under the Lie bracket, it is a CR structure [4]. The CR condition is equivalent to

$$\mathcal{S}_A(X, Y) = [AX, AY] + A[AX, A^2Y] + A[A^2X, AY] - [A^2X, A^2Y] = 0. \quad (1.9)$$

Indeed, (1.9) holds on arguments that are eigenvectors iff  $H$  is closed under the Lie bracket. If (1.9) holds, we will say that  $A$  is an *F structure of the CR type*.

A *generalized F structure* is defined by an endomorphism  $\Phi$  of  $\mathbf{T}M$ , which is  $g$ -skew-symmetric and satisfies the condition  $\Phi^3 + \Phi = 0$ . Thus, the eigenvalues of  $\Phi$  are  $\pm i, 0$ . We may represent  $\Phi$  by the right hand side of (1.3) but, the conditions on the tensor fields  $A, \pi, \sigma$  will be different [17]. Equivalently, the structure may be defined by the  $\pm i$ -eigenbundles  $E, \bar{E}$  and the 0-eigenbundle  $S$ , where  $E$  is a complex  $g$ -isotropic (possibly not maximal) subbundle of  $\mathbf{T}^cM$  such that  $E \cap \bar{E}^{\perp_g} = 0$  [17]. Notice that  $S^c = E^{\perp_g} \cap \bar{E}^{\perp_g}$  and we have the decomposition  $\mathbf{T}^cM = E \oplus \bar{E} \oplus S^c$ . The projections of  $\mathbf{T}^cM$  on  $E, \bar{E}, S^c$  are defined by formulas (1.8) where  $A$  is replaced by  $\Phi$ .

The structure defined by  $\Phi$  is said to be integrable and, then, it is called a *generalized CRF structure* if the subbundle  $E$  is closed under Courant brackets. As in the classical case, by looking at arguments that belong to the various eigenbundles of  $\Phi$ , we see that the integrability condition is equivalent to  $\mathcal{S}_\Phi((X, \alpha), (Y, \beta)) = 0$ , where  $\mathcal{S}_\Phi$  is defined by (1.9) with  $A$  replaced by  $\Phi$  and Lie brackets replaced by Courant brackets [17].  $C^\infty(M)$ -bilinearity of  $\mathcal{S}_\Phi$  follows from  $\Phi^3 + \Phi = 0$ .

If a generalized F structure  $\Phi$  preserves the tangent and cotangent bundle of  $M$ , we have  $\pi = 0, \sigma = 0$  and  $\Phi$  may be identified with a classical F structure  $A$ . Then, if  $\Phi$  is integrable  $A$  is called a *classical CRF structure*. The conditions that characterize a classical CRF structure are stronger than the demand that the  $i$ -eigenbundle  $H$  of  $A$  is CR, namely, these conditions are [17],

$$[H, H] \subseteq H, \quad [H, Q^c] \subseteq H \oplus Q^c, \quad (1.10)$$

equivalently,

$$\begin{aligned} \mathcal{N}_A(X, Y) &= pr_Q[X, Y], \quad \forall X, Y \in P, \\ \mathcal{N}_A(X, Y) &= 0, \quad \forall X \in P, Y \in Q. \end{aligned} \quad (1.11)$$

In [17] we also discussed generalized F structures  $\Phi$  with classical square, i.e., such that  $\Phi^2$  preserves the tangent and cotangent bundle of  $M$ . These are characterized by the conditions

$$A \circ \sharp_\pi = \sharp_\pi \circ A^*, \quad \flat_\sigma \circ A = A^* \circ \flat_\sigma, \quad (1.12)$$

equivalently,

$$\pi_A(\alpha, \beta) = \pi(A^* \alpha, \beta), \quad \sigma_A(X, Y) = \sigma(AX, Y) \quad (1.13)$$

define a bivector and a 2-form, respectively.

The aim of this paper is to study the generalized CRF structures introduced by the following definition, which was suggested by the generalized complex structures equivalent to holomorphic Poisson structures.

**Definition 1.1.** A generalized F structure  $\Phi$  such that  $\Phi(TM) \subseteq TM$  and  $\Phi^2(T^*M) \subseteq T^*M$  will be called a *quasi-classical generalized F structure*. If integrable, the structure will be called a *quasi-classical generalized CRF structure*.

In Section 2, we describe the matrix of tensor components of a quasi-classical generalized F structure and we get the integrability conditions of the structure. These include the fact that the endomorphism  $A$  is a classical CRF structure and the bivector field  $\pi$  is a Poisson structure. Some conditions that relate between  $A$  and  $\pi$  must also be added. In Section 3, we discuss the case where one of the subbundles  $P, Q$  defined by the endomorphism  $A$  is a foliation. In the last section we show that the Poisson structure  $\pi$  induces a Lie algebroid structure on the dual bundle  $P^*$ . Then, we define a spectral sequence that converges to the Poisson cohomology of  $\pi$  and we show that second term of this sequence has a Dolbeault type grading.

## 2 Quasi-classical F and CRF structures

We begin by characterizing the quasi-classical generalized F structures.

**Proposition 2.1.** A quasi-classical generalized F structure  $\Phi$  is equivalent with a pair  $(A, \pi)$  where  $A$  is an F structure and  $\pi$  a bivector field that is  $A$ -compatible in the sense that  $\pi_A$  is again a bivector and  $\text{im } \sharp_\pi \subseteq \text{im } A = P$ .

*Proof.* Definition 1.1 implies that a quasi-classical generalized F structure  $\Phi$  has a classical square and that the following conditions hold

$$\sigma = 0, \quad A \circ \sharp_\pi = \sharp_\pi \circ A^* \quad (2.1)$$

(see (1.12)). The second condition (2.1) is equivalent to the fact that  $\pi_A$  defined by (1.13) is a bivector field.

Accordingly, the matrix representation of the structure has the form

$$\Phi \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \sharp_\pi \\ 0 & -A^* \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}, \quad (2.2)$$

equivalently,  $\Phi(X, \alpha) = (AX + \sharp_\pi \alpha, -A^* \alpha)$ . Furthermore,  $\Phi^3 + \Phi = 0$  is equivalent to

$$A^3 + A = 0, A \circ (\sharp_\pi \circ A^* - A \circ \sharp_\pi) = \sharp_\pi \circ (A^{*2} + Id) \quad (2.3)$$

(look at the cases  $\alpha = 0$  and  $X = 0$ ).

Thus, firstly,  $A$  must be a classical F-structure and, secondly, since  $im(A^{*2} + Id) = ann P$  and (2.1) holds, the last condition (2.3) becomes  $\sharp_\pi(ann P) = 0$ . Finally, the relation

$$\langle \beta, \sharp_\pi \alpha \rangle = -\langle \alpha, \sharp_\pi \beta \rangle = 0, \alpha \in T^*M, \beta \in ann P \quad (2.4)$$

shows that  $\sharp_\pi(ann P) = 0 \Leftrightarrow im(\sharp_\pi) \subseteq P$ .  $\square$

**Remark 2.1.** The endomorphism  $\Phi$  vanishes on  $Q \oplus ann P$  ( $ann P \approx Q^*$ ) and a dimension argument tells us that the 0-eigenbundle of  $\Phi$  is  $S = Q \oplus ann P$ .

**Proposition 2.2.** *Consider the quasi-classical generalized F structure defined by  $(A, \pi)$  and let  $H, \bar{H}, Q$  be the  $\pm i, 0$ -eigenbundles of  $A$ . Then, if  $(h_i), (\bar{h}_i), (q_j)$  are local bases of  $H, \bar{H}, Q$ , respectively, the local expression of  $\pi$  must be of the form*

$$\pi = \frac{1}{2}(\pi^{ij} h_i \wedge h_j + \bar{\pi}^{ij} \bar{h}_i \wedge \bar{h}_j) \quad (\pi^{ij} + \pi^{ji} = 0). \quad (2.5)$$

*Proof.* The decomposition  $T^c M = H \oplus \bar{H} \oplus Q^c$  dualizes to  $T^{*c} M = H^* \oplus \bar{H}^* \oplus Q^{*c}$ , where the terms are the  $\pm i, 0$ -eigenbundles of  $A^*$  and we may identify

$$H^* = ann(\bar{H} \oplus Q^c), \bar{H}^* = ann(H \oplus Q^c), Q^* = ann P.$$

Since  $\sharp_\pi(ann P) = 0$ , the local expression of  $\pi$  has no terms containing  $q_i$ . Furthermore, for  $\nu \in H^*, \bar{H}^*, A \sharp_\pi \nu = \sharp_\pi(A^* \nu) = \pm i \sharp_\pi \nu$ . Hence,  $\sharp_\pi : H^* \rightarrow H, \sharp_\pi : \bar{H}^* \rightarrow \bar{H}$  and  $\pi$  can take non-zero values only if evaluated on two arguments that are both either in  $H^*$  or in  $\bar{H}^*$ . Therefore,  $\pi$  has no term  $\pi^{ij} h_i \wedge \bar{h}_j$  and, since  $\pi$  is real, we have the required conclusion. (In (2.5) and in the whole paper we use the Einstein summation convention.)  $\square$

**Proposition 2.3.** *The  $i$ -eigenbundle  $E$  of the quasi-classical generalized F structure  $\Phi$  is given by the formula*

$$E = \{(Z - \frac{1}{2} \sharp_\pi \xi, \xi) / Z \in H, \xi \in \bar{H}^* = ann(H \oplus Q)\}. \quad (2.6)$$

*Proof.* Using (2.2), we may write  $E = E_P + E_Q$ , where

$$E_P = \{(X, \alpha) \in \mathbf{T}^c M / X \in P^c, AX + \sharp_\pi \alpha = iX, -A^* \alpha = i\alpha\}$$

$$E_Q = \{(Y, \beta) \in \mathbf{T}^c M / Y \in Q^c, \sharp_\pi \beta = iY, -A^* \beta = i\beta\}.$$

In  $E_Q$ , since  $im \sharp_\pi \subseteq P$ , we have  $Y = 0, \sharp_\pi \beta = 0$ , therefore,  $E_Q \subseteq \bar{H}^* \subseteq E_P$  and  $E = E_P$ . Furthermore, (1.8) implies

$$E = E_P = \{(X, \alpha) / X \in P^c, \alpha \in \bar{H}^* = ann(H \oplus Q), \sharp_\pi \alpha = -2pr_{\bar{H}} X\}.$$

Formula (2.6) is a reformulation of this result.  $\square$

The integrability conditions of a quasi-classical generalized F structure are given by the following theorem.

**Theorem 2.1.** *The quasi-classical generalized F structure  $\Phi$  defined by the pair of tensor fields  $(A, \pi)$  is a quasi-classical generalized CRF structure iff (1) the endomorphism  $A$  is a classical CRF structure, (2)  $\pi$  is a Poisson structure, (3)  $R_{(\pi, A)}(X, \beta) = 0$  for  $X \in P, \beta \in \text{ann } Q$  ( $P = \text{im } A, Q = \text{ker } A$ ).*

*Proof.* The tensor fields  $A, \pi$  satisfy the conditions stated in Proposition 2.1 and we refer to the eigenbundles of  $A$  in the notation below. If  $\Phi$  is quasi-classical, the condition  $\mathcal{S}_\Phi((X, 0), (Y, 0)) = 0$  becomes  $\mathcal{S}_A(X, Y) = 0$ , hence, it is equivalent with the fact that  $A$  is a structure of the CR type.

Obviously, it suffices to look at arguments of the form  $(X, 0)$  and  $(0, \alpha)$  separately. The arguments  $(X, 0), (Y, 0)$  were already considered and we need to compute  $\mathcal{S}_\Phi((0, \alpha), (0, \beta))$  and  $\mathcal{S}_\Phi((X, 0), (0, \beta))$ . Moreover, since  $\mathcal{S}_\Phi$  vanishes if one of the arguments belongs to  $S = Q \oplus (\text{ann } P)$  (see Remark 2.1), it suffices to establish the integrability conditions for  $(X, 0), X \in P$  and  $(0, \beta), \beta \in \text{ann } Q$ .

Computing the Courant brackets involved and using the following consequences of (2.1)

$$\pi(\alpha \circ A) = \pi(\alpha, \beta \circ A), \quad \pi(\alpha \circ A^2) = \pi(\alpha, \beta \circ A^2),$$

we get

$$\begin{aligned} \mathcal{S}_\Phi((0, \alpha), (0, \beta)) &= ([\sharp_\pi \alpha, \sharp_\pi \beta] + \sharp_\pi [L_{\sharp_\pi \alpha}(\beta \circ A^2) - L_{\sharp_\pi \beta}(\alpha \circ A^2) \\ &\quad - d(\pi(\alpha \circ A^2, \beta))], L_{\sharp_\pi \beta}(\alpha \circ A) - L_{\sharp_\pi \alpha}(\beta \circ A) - d(\pi(\alpha \circ A, \beta)) \\ &\quad + [L_{\sharp_\pi \alpha}(\beta \circ A^2) - L_{\sharp_\pi \beta}(\alpha \circ A^2) - d(\pi(\alpha \circ A^2, \beta))] \circ A) \end{aligned} \quad (2.7)$$

For  $\alpha, \beta \in \text{ann } Q$ , we have  $\alpha \circ A^2 = -\alpha, \beta \circ A^2 = -\beta$  and the vanishing of the vector part of (2.7) yields the Poisson condition

$$[\sharp_\pi \alpha, \sharp_\pi \beta] = \sharp_\pi \{\alpha, \beta\}_\pi \quad (\alpha, \beta \in \text{ann } Q), \quad (2.8)$$

where

$$\{\alpha, \beta\}_\pi = L_{\sharp_\pi \alpha} \beta - L_{\sharp_\pi \beta} \alpha - d(\pi(\alpha, \beta)) \quad (2.9)$$

is the Poisson bracket of 1-forms [13].

Since  $\forall \alpha, \beta, \gamma \in T^*M$  the Gelfand-Dorfman formula [6]

$$[\pi, \pi](\alpha, \beta, \gamma) = 2[\gamma(\sharp_\pi \{\alpha, \beta\}_\pi - [\sharp_\pi \alpha, \sharp_\pi \beta]) \quad (2.10)$$

holds, condition (2.8) is equivalent to

$$[\pi, \pi](\alpha, \beta, \gamma) = 0, \quad \forall \alpha, \beta \in \text{ann } Q, \gamma \in T^*M. \quad (2.11)$$

But, we can show that (2.11) holds iff  $\pi$  is a Poisson bivector, in other words,  $[\pi, \pi] = 0$ , equivalently (2.8), hold for any arguments. Indeed, since  $[\pi, \pi]$  is totally skew-symmetric, (2.11) means that  $[\pi, \pi] = 0$  whenever at least two

arguments belong to  $\text{ann } Q$ . On the other hand, if at least two arguments, e.g.,  $\alpha, \beta \in \text{ann } P$ , then,  $\sharp_\pi \alpha = 0, \sharp_\pi \beta = 0$  and  $\{\alpha, \beta\}_\pi = 0$  (see (2.9)), therefore,  $[\pi, \pi] = 0$  because of (2.10). Since this covers all the possible cases in the decomposition  $T^*M = (\text{ann } Q) \oplus (\text{ann } P)$ , we are done.

Furthermore, the vanishing of the covector part of (2.7) is the condition

$$\{\alpha, \beta\}_\pi \circ A = L_{\sharp_\pi \alpha}(\beta \circ A) - L_{\sharp_\pi \beta}(\alpha \circ A) - d(\pi(\alpha \circ A, \beta)), \quad \alpha, \beta \in \text{ann } Q, \quad (2.12)$$

which is known for generalized complex structures (e.g., [15]). The insertion of the value of  $\{\alpha, \beta\}_\pi$  given by (2.9) in (2.12) shows that the latter is equivalent to the vanishing of the bivector field (see [15])

$$C_{(\pi, A)}(\alpha, \beta) = \beta \circ L_{\sharp_\pi \alpha} A - \alpha \circ L_{\sharp_\pi \beta} A + d(\pi(\alpha, \beta)) \circ A - d(\pi(\alpha \circ A, \beta)) = 0. \quad (2.13)$$

Now, we shall discuss the condition  $\mathcal{S}_\Phi((X, 0), (0, \beta)) = 0$  for  $X \in P, \beta \in \text{ann } Q$ , i.e.,  $A^2 X = -X, \beta \circ A^2 = -\beta$  and

$$\begin{aligned} \Phi(X, 0) &= (AX, 0), \quad \Phi^2(X, 0) = -(X, 0), \\ \Phi(0, \beta) &= (\sharp_\pi \beta, -\beta \circ A), \quad \Phi^2(0, \beta) = -(0, \beta). \end{aligned}$$

The required value of  $\mathcal{S}_\Phi$  is

$$\begin{aligned} \mathcal{S}_\Phi((X, 0), (0, \beta)) &= ([AX, \sharp_\pi \beta] - A[X, \sharp_\pi \beta] + \sharp_\pi [L_X(\beta \circ A) - L_{AX} \beta], \\ &\quad - L_{AX}(\beta \circ A) - L_X \beta - [L_X(\beta \circ A) - L_{AX} \beta] \circ A). \end{aligned}$$

The vector component of the previous expression vanishes iff

$$(L_{\sharp_\pi \beta} A)(X) = \sharp_\pi [L_X(\beta \circ A) - L_{AX} \beta] \Leftrightarrow R_{(\pi, A)}(\beta, X) = 0, \quad (2.14)$$

where  $R_{(\pi, A)}$  is the *Schouten concomitant*. Due to the general relation [12]

$$\alpha(R_{(\pi, A)}(\beta, X)) = \langle C_{(\pi, A)}(\alpha, \beta), X \rangle,$$

this condition is equivalent to (2.13) and only one of them, e.g., (2.14) must be required.

Finally, the covector component of the condition  $\mathcal{S}_\Phi((X, 0), (0, \beta)) = 0$  is

$$[L_{AX} \beta - L_X(\beta \circ A)] \circ A = L_{AX}(\beta \circ A) + L_X \beta, \quad (X \in P, \beta \in \text{ann } Q) \quad (2.15)$$

and it splits into the cases (i)  $AX = iX, \beta \circ A = i\beta$ , (ii)  $AX = iX, \beta \circ A = -i\beta$  and conjugates. In case (i) the condition holds trivially. In case (ii) the condition becomes  $(L_X \beta) \circ A = -i(L_X \beta)$ , equivalently,

$$L_X \beta \in \text{ann}(H \oplus Q^c), \quad \forall X \in H, \beta \in \text{ann}(H \oplus Q^c). \quad (2.16)$$

Condition (2.16) means  $\langle L_X \beta, Z \rangle = 0$  for  $Z \in H \oplus Q^c$ , which is equivalent to  $[X, Z] \in H \oplus Q^c, \forall X \in H, Z \in Q^c$ . Together with the condition  $[H, H] \subseteq H$ , which holds because  $\mathcal{S}_A = 0$ , we have (1.10), therefore,  $A$  is a classical CRF structure.

The conclusion of the theorem is the sum of the conditions deduced during the proof.  $\square$

Notice that, even in the integrable case, condition (3) may not hold for other kind of arguments  $(X, \beta)$ .

**Remark 2.2.** We know that (1.10) is equivalent to (1.11). For any arguments, the Nijenhuis tensor of  $A$  is given by

$$\begin{aligned}\mathcal{N}_A(X, Y) &= [AX, AY] - A[AX, Y] - A[X, AY] + A^2[X, Y] \\ &= [AX, AY] - A[AX, Y] - A[X, AY] - [X, Y] \\ &\quad + pr_Q[X, Y].\end{aligned}\tag{2.17}$$

Therefore, the first condition (1.11) is equivalent to

$$[AX, AY] - A[AX, Y] - A[X, AY] - [X, Y] = 0, \quad \forall X, Y \in P,$$

which is equivalent to  $[H, H] \subseteq H$  (check this on  $\pm i$ -eigenvectors  $X, Y$ ). Then, if we take  $X \in P, Y \in Q$ , (2.17) shows that the second condition (1.11) takes the form

$$A^2[X, Y] - A[AX, Y] = 0, \quad \forall X \in P, Y \in Q.\tag{2.18}$$

Thus,  $A$  is a classical CRF structure iff  $[H, H] \subseteq H$  and (2.18) holds.

**Proposition 2.4.** *In Theorem 2.1, condition (3) may be replaced by the pair of conditions*

$$[\sharp_\pi \beta, X] \in P, \quad (L_Y \pi)(\lambda, \mu) = 0,\tag{2.19}$$

where  $X \in P, Y \in \bar{H}, \beta \in \text{ann } Q, \lambda, \mu \in \text{ann}(\bar{H} \oplus Q^c)$ .

*Proof.* We will replace condition (3) of Theorem 2.1, written under the form (2.14) by its evaluation on 1-forms  $\lambda$ . For  $\lambda \in \text{ann } P$ , (2.14) becomes  $(L_{\sharp_\pi \beta} A)(X) = 0$ , which is equivalent to

$$\langle \lambda, [\sharp_\pi \beta, AX] \rangle = 0, \quad \forall X \in P, \lambda \in \text{ann } P.\tag{2.20}$$

This is equivalent to the first condition (2.19) and it means that the Hamiltonian vector fields  $\sharp_\pi \beta$  preserve the distribution  $P$ . The tensorial character of this condition follows from  $X \in P, \beta \in \text{ann } Q$ .

For  $\lambda \in \text{ann } Q^c = H^* \oplus \bar{H}^*$  ( $H^* = \text{ann}(\bar{H} \oplus Q^c)$ ,  $\bar{H}^* = \text{ann}(H \oplus Q^c)$ ), since  $X \in P^c = H \oplus \bar{H}$ , the evaluation splits into the cases: (i)  $AX = iX$ ,  $A^* \beta = i\beta$ , (ii)  $AX = iX$ ,  $A^* \beta = -i\beta$  and their complex conjugates and the results are

$$(i) \quad A[\sharp_\pi \beta, X] - i[\sharp_\pi \beta, X] = 0, \quad (ii) \quad A[\sharp_\pi \beta, X] - i[\sharp_\pi \beta, X] = 2i\sharp_\pi(L_X \beta).$$

Condition (i) holds because  $\sharp_\pi \beta, X \in H$  and  $H$  is closed by brackets since  $A$  is of the CR type. Condition (ii) again splits into two cases (a)  $\lambda \in H^*$ , (b)  $\lambda \in \bar{H}^*$ . In case (a) we have  $\lambda \circ A = i\lambda$  and the evaluation of  $\lambda$  on the left hand side of (ii) gives zero. The same holds for the right hand side:

$$\langle \lambda, \sharp_\pi(L_X \beta) \rangle = - \langle L_X \beta, \sharp_\pi \lambda \rangle = -X(\pi(\lambda, \beta)) + \langle \beta, [X, \sharp_\pi \lambda] \rangle = 0,$$



because of (2.5) and of  $[X, \sharp_\pi \lambda] \in H$ . But, in case (b)  $\lambda \circ A = -i\lambda$  and the left hand side of (ii) becomes

$$-2i < \lambda, [\sharp_\pi \beta, X] > = 2i(< \lambda, \sharp_{L_X \pi} \beta + \sharp_\pi(L_X \beta) >).$$

Thus, (ii) becomes

$$(L_X \pi)(\beta, \lambda) = 0, \quad \forall \beta, \lambda \in \text{ann}(H \oplus Q^c). \quad (2.21)$$

This condition is tensorial in  $X$  since

$$(L_{fX} \pi)(\beta, \lambda) = f(L_X \pi)(\beta, \lambda) - \beta(X)\pi(df, \lambda) - \lambda(X)\pi(\beta, df)$$

and  $X \in H$  implies  $\beta(X) = 0, \lambda(X) = 0$ . By conjugation, (2.21) yields the second condition (2.19).  $\square$

**Corollary 2.1.** *Let  $(A_u, \pi_u)$  define quasi-classical generalized CRF structures on manifolds  $M_u$ ,  $u = 1, 2$ . Then,  $(A_1 + A_2, \pi_1 + \pi_2)$  defines a quasi-classical generalized CRF structure on  $M_1 \times M_2$ .*

*Proof.* Checking the conditions of Proposition 2.1 and Theorem 2.1 is straightforward.  $\square$

**Remark 2.3.** We shall indicate the following equivalent expressions of the first condition (2.19). This condition is equivalent to  $L_{\sharp_\pi \beta} A^2(X) = 0, \forall X \in P, \forall \beta \in \text{ann } Q$ . On the other hand, the form (2.20) of the condition is equivalent to  $d\lambda(\sharp_\pi \beta, X) = 0$  and, then, to  $L_{\sharp_\pi \beta} \lambda \in \text{ann } P, \forall \lambda \in \text{ann } P$ .

### 3 Quasi-classical structures with foliations

The simplest example of a quasi-classical generalized CRF structure is that of a locally product structure defined by two involutive subbundles  $P, Q \subseteq TM$  (foliations), where the leaves of  $P$  are endowed with a holomorphic Poisson structure ( $A_P \in \text{End } P, \pi_P \in \wedge^2 P$ ). Then, we also have  $\pi \in \chi^2(M)$  and, if we extend  $A_P$  to  $A \in \text{End}(TM)$  by  $A|_Q = 0$ , it is easy to check all the conditions stated in Proposition 2.1 and Theorem 2.1, respectively, Proposition 2.4. (It suffices to use vector fields in  $P$ , respectively  $Q$ , with local components that depend only on the coordinates on the leaves of  $P$ , respectively  $Q$ .)

Below, we shall discuss quasi-classical generalized CRF structures where either  $Q$  or  $P$  is a foliation.

**Proposition 3.1.** *Let  $\Phi$  be a quasi-classical generalized CRF structure defined by a pair  $(A, \pi)$  such that  $Q = \ker A$  defines a foliation and the bivector field  $\pi$  is projectable to the local transversal submanifolds of the leaves of  $Q$ . Then,  $A$  induces a transversal holomorphic structure of  $Q$  and  $\pi$  projects to a  $Q$ -transversal holomorphic Poisson structure.*

*Proof.* First, we will show that an F structure  $A$  of the CR type with an integrable 0-eigenbundle  $Q$  is a classical CRF structure iff the tensor field  $A$  is projectable to the local transversal submanifolds of the leaves of  $Q$ . Indeed, if  $A$  is classical CRF, we have (2.18) (Remark 2.2). If we assume there that  $X \in P$  is a  $Q$ -projectable vector field, equivalently,  $\forall Y \in Q, [X, Y] \in Q$ , (2.18) becomes  $A[AX, Y] = 0$ , whence  $[AX, Y] \in Q$  and  $AX$  is projectable too. This exactly is what projectability of  $A$  means. Conversely, if  $A$  and  $X$  are projectable,  $[X, Y], [AX, Y] \in Q$  and (2.18) holds for a projectable  $X$ . It is easy to check that this implies (2.18) for  $X$  replaced by  $fX$ , where  $f$  is an arbitrary function, i.e., (2.18) holds for any arguments. Notice that, if it exists, the projection  $\tilde{A}$  of  $A$  to the local  $Q$ -transversal submanifolds is given by  $\tilde{A}[X]_Q = [AX]_Q$ , where  $X$  is projectable and the index  $Q$  denotes the image in the quotient space.

In the proposition,  $\Phi$  is a quasi-classical generalized CRF structure, hence,  $A$  is classical CRF and the induced tensor field  $\tilde{A}$  exists. For any projectable  $X$ , we have

$$\tilde{A}^2[X]_Q = [A^2X]_Q = [(A^2 + Id)(X) - X]_Q = -[X]_Q,$$

which means that  $\tilde{A}$  is almost complex. Let us compute the Nijenhuis tensor  $\mathcal{N}_{\tilde{A}}$  by using foliated vector fields  $X, Y \in P$ . We get

$$\begin{aligned} \mathcal{N}_{\tilde{A}}([X]_Q, [Y]_Q) &= [\tilde{A}[X]_Q, \tilde{A}[Y]_Q] - \tilde{A}[\tilde{A}[X]_Q, [Y]_Q] \\ &\quad - \tilde{A}[[X]_Q, \tilde{A}[Y]_Q] - [[X]_Q, [Y]_Q] \\ &= [[AX, AY] - A[AX, Y] - A[X, AY] - [X, Y]]_Q \\ &= [\mathcal{N}_A(X, Y) - pr_Q[X, Y]]_Q = 0. \end{aligned}$$

The first equality holds because the Lie bracket is compatible with the projection to the  $Q$ -transversal submanifolds and the last equality holds because  $H$  is involutive (see Remark 2.2). This proves the existence of the transversal holomorphic structure of  $Q$ .

Now, we look at the Poisson bivector field  $\pi$  of the CRF structure  $\Phi$ . Since  $Q$  is a transversally holomorphic foliation, each point of  $M$  has a coordinate neighborhood with local coordinates  $(z^i, y^u)$ , where  $z^i$  are lifts of complex coordinates defined by  $\tilde{A}$  on the local transversal submanifolds of  $Q$  and  $y^u$  are real coordinates on the leaves of  $Q$  (e.g., see [5]). These produce local bases of  $TM$  of the form

$$Z_i = \frac{\partial}{\partial z_i} - t_i^u \frac{\partial}{\partial y^u} \in H, \bar{Z}_i = \frac{\partial}{\partial \bar{z}_i} - \bar{t}_i^u \frac{\partial}{\partial y^u} \in \bar{H}, Y_u = \frac{\partial}{\partial y^u},$$

where  $t_i^u$  are some local functions of  $(z, \bar{z}, y)$ . Thus, the expression (2.5) becomes

$$\pi = \frac{1}{2}(\pi^{ij}(z, \bar{z})Z_i \wedge Z_j + \bar{\pi}^{ij}(z, \bar{z})\bar{Z}_i \wedge \bar{Z}_j).$$

The local coefficients  $\pi^{ij}$  do not depend on  $y$  because of the hypothesis that  $\pi$  is  $Q$ -projectable. (The projectability of  $\pi$  means the existence of a bivector field

$\tilde{\pi}$  on the local transversal submanifolds of the leaves of  $Q$  such that  $\pi$  and  $\tilde{\pi}$  are related by the natural projection onto the submanifold, which happens iff  $\pi^{ij}$  do not depend on  $y$ . The  $Q$ -projectability of  $\pi$  is also equivalent to  $\text{im } \sharp_\pi \subseteq P$  together with the fact that, locally,  $\forall Y \in Q$ ,  $L_Y \pi$  belongs to the ideal generated by the tangent vector fields of the leaves of  $Q$ .)

Furthermore,  $\pi$  satisfies conditions (2.19), in particular,

$$(L_Y \pi)(\lambda, \mu) = 0, \quad \forall Y \in \bar{H}, \lambda, \mu \in \text{ann}(\bar{H} \oplus Q^c).$$

For  $Y = \bar{Z}_i$ ,  $\lambda = dz^h$ ,  $\mu = dz^k$ , this yields  $\bar{Z}_i(\pi^{hk}) = 0$ .

Thus, the projection  $\tilde{\pi}$  of  $\pi$  is

$$\tilde{\pi} = \frac{1}{2}(\pi^{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \bar{\pi}^{ij}(\bar{z}) \frac{\partial}{\partial \bar{z}_i} \wedge \frac{\partial}{\partial \bar{z}_j}).$$

Finally, the condition  $[\tilde{\pi}, \tilde{\pi}] = 0$  holds because, like the Lie bracket, the Schouten-Nijenhuis bracket is compatible with the projection onto the local transversal submanifolds of the leaves of  $Q$ . Therefore,  $\tilde{\pi}$  is a holomorphic Poisson structure on the local  $Q$ -transversal submanifolds, endowed with the complex structure  $\bar{A}$ .  $\square$

**Proposition 3.2.** *Let  $\Phi$  be a quasi-classical generalized  $F$  structure defined by the pair  $(A, \pi)$  where the tangent subbundle  $P = \text{im } A$  is a foliation. Then,  $\Phi$  is integrable (i.e., CRF) iff (i) the pair  $(A|_P, \pi|_{\text{ann } Q})$  ( $Q = \ker A$ ) defines holomorphic Poisson structures on the leaves of  $P$ , (ii)  $\mathcal{N}_A(X, Y) = 0 \quad \forall X \in P, Y \in Q$ .*

*Proof.* Since  $P = \text{im } A$  and  $\text{ann } Q = P^*$ ,  $(A|_P, \pi|_{\text{ann } Q})$  defines quasi-classical generalized, almost complex structures of the leaves of  $P$  (see Introduction). If  $\Phi$  is integrable,  $A$  is a classical CRF structure and (1.11) implies (ii) of the proposition. On the other hand, the first condition (1.11) and formula (2.17) imply  $\mathcal{N}_{A|_P} = 0$ . Therefore,  $A$  induces complex analytic structures on the leaves of  $P$  and  $M$  is covered by local charts with real coordinates  $(x^a)$  and complex coordinates  $(z^i, \bar{z}^i)$  such that  $P$  is defined by  $dx^a = 0$  and  $(z^i, \bar{z}^i)$  are complex analytic coordinates along the leaves of  $P$ . Accordingly, the 0-eigenbundle  $Q^c$  of  $A$  and  $\text{ann } Q^c$  have local bases of the form

$$X_a = \frac{\partial}{\partial x^a} - t_a^i \frac{\partial}{\partial z^i} - \bar{t}_a^{\bar{i}} \frac{\partial}{\partial \bar{z}^{\bar{i}}}, \quad \theta^i = dz^i + t_a^i dx^a, \quad \bar{\theta}^{\bar{i}} = d\bar{z}^{\bar{i}} + \bar{t}_a^{\bar{i}} dx^a.$$

Furthermore, the Poisson bivector field  $\pi$ , which, in view of (2.5), has the local expression

$$\pi = \frac{1}{2}(\pi^{il}(z, \bar{z}, x) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_l} + \bar{\pi}^{i\bar{l}}(z, \bar{z}, x) \frac{\partial}{\partial \bar{z}_i} \wedge \frac{\partial}{\partial \bar{z}_l}), \quad (3.1)$$

induces Poisson structures  $\pi|_{\text{ann } Q}$  on the leaves of  $P$  that have the same local expressions, but, with  $x = \text{const}$ . Moreover, the second integrability condition (2.19) for  $\Phi$  reduces to

$$(L_{\frac{\partial}{\partial \bar{z}^{\bar{i}}}} \pi)(\theta^h, \theta^k) = \frac{\partial \pi^{h,k}}{\partial \bar{z}^{\bar{i}}} = 0,$$

which means that  $\pi$  is holomorphic along the leaves. Thus, integrability of  $\Phi$  also implies (i).

Conversely, it is clear that condition (i) implies the existence of the local coordinates  $(x^a, z^i, \bar{z}^i)$  described above and the local expression (3.1), where  $\partial\pi^{hk}/\partial\bar{z}^i = 0$ . This local expression implies  $[\pi, \pi] = 0$ , hence,  $\pi$  is a Poisson bivector field. The fact that  $A$  is classical CRF is ensured by the integrability of  $A|_P$  together with condition (ii), which are exactly the two conditions (1.11) in our case. Finally, the first condition (2.19) is a consequence of the involutivity of  $P$  and the second condition (2.19) holds since we have  $\partial\pi^{hk}/\partial\bar{z}^i = 0$ . Thus, all the conditions of Theorem 2.1 and Proposition 2.4 hold and  $\Phi$  is integrable.  $\square$

**Remark 3.1.** If  $\Phi$  is a quasi-classical generalized CRF structure such that the corresponding tensor fields satisfy the condition  $im \sharp_\pi = im A$ , then,  $P = im A$  is the symplectic foliation of the Poisson structure  $P$  and  $(A|_P, \pi|_{ann Q})$  define holomorphic symplectic structures on the leaves of  $P$ .

We exemplify Proposition 3.2 by the following structures.

A generalized almost contact structure of codimension  $h$  is a system of tensor fields  $(A \in End(TM), Z_a \in \chi(M), \pi \in \chi^2(M), \sigma \in \Omega^2(M), \xi^a \in \Omega^1(M))$  ( $\chi(M) = \chi^1(M)$ ) that satisfies the following conditions [16]

$$\begin{aligned} \pi(\alpha \circ A, \beta) &= \pi(\alpha, \beta \circ A), \quad \sigma(AX, Y) = \sigma(X, AY), \\ A(Z_a) &= 0, \quad \xi^a \circ A = 0, \quad i(Z_a)\sigma = 0, \quad i(\xi^a)\pi = 0, \quad \xi^a(Z_b) = \delta_b^a, \\ A^2 &= -Id - \sharp_\pi \circ \flat_\sigma + \sum_{a=1}^h \xi^a \otimes Z_a. \end{aligned} \quad (3.2)$$

Furthermore, the structure is *normal* if [16]:

$$\begin{aligned} [\pi, \pi] &= 0, \quad R_{(\pi, A)} = 0, \\ L_{Z_a}\pi &= 0, \quad L_{Z_a}\sigma = 0, \quad L_{\sharp_\pi \alpha} \xi^a = 0, \\ \mathcal{N}_A(X, Y) &= \sharp_\pi(i(X \wedge Y)d\sigma) - \sum_{a=1}^h (d\xi^a(X, Y))Z_a, \\ d\sigma_A(X, Y, Z) &= \sum_{Cycl(X, Y, Z)} d\sigma(AX, Y, Z) \\ [Z_a, Z_b] &= 0, \quad L_{Z_b}\xi^a = 0, \quad L_{Z_a}A = 0, \\ (L_{AX}\xi^a)(Y) - (L_{AY}\xi^a)(X) &= 0. \end{aligned} \quad (3.3)$$

For a generalized almost contact structure of codimension  $h$  such that  $\sigma = 0$ , conditions (3.2) imply the fact that the pair  $(A, \pi)$  defines a quasi-classical generalized F structure  $\Phi$ . Since  $-1$  is not an eigenvalue of  $\sharp_\pi \circ \flat_\sigma = 0$ , the normality of the structure  $(A, \pi, 0, Z_a, \xi^a)$  is characterized by the first four lines of (3.3) [16]. It is easy to see that these conditions imply the integrability of  $\Phi$ . In particular, if  $X, Y \in P = ann \{\xi^a\}$  and since  $Q = span\{Z_a\}$ , we have

$$\mathcal{N}_A(X, Y) = - \sum_a d\xi^a(X, Y)Z_a = \sum_a \xi^a([X, Y])Z_a = pr_Q[X, Y]. \quad (3.4)$$

Furthermore, if  $X \in P, Y \in Q$ ,

$$\begin{aligned}\mathcal{N}_A(X, Y) &= -\sum_a d\xi^a(X, Y)Z_a = \sum_a (L_X \xi^a)(Y) \\ &= \sum_a (L_{AX'} \xi^a)(Y) = \sum_a (L_{AY} \xi^a)(X) = 0,\end{aligned}$$

where  $X = AX'$  (we use  $P = \text{im } A$ ) and the key fourth equality sign is given by the last line of (3.3) (of course,  $AY = 0$  because  $Y \in Q$ ). These results about  $\mathcal{N}_A$  show that  $A$  is a classical CRF structure. Condition  $[Z_a, Z_b] = 0$  included in (3.3) shows that the 0-eigenbundle  $Q = \text{span}\{Z_a\}$  is a foliation.

The geometric structure of a normal generalized contact structure of codimension  $h$  was described in Theorem 3.3 of [16] and includes the  $Q$ -transversal holomorphic Poisson structure obtained in Proposition 3.1. In particular, Example 3.2 of [16] tells that, if  $(N, A, \pi)$  is a holomorphic Poisson manifold and  $M$  is a principal torus bundle over  $N$  endowed with a connection  $\xi$  of curvature form  $\Xi$ , then, the conditions (a)  $i(\sharp_{\pi^h} \alpha)\Xi = 0$ , (b)  $\Xi((A^h X, A^h Y)) = \Xi(X, Y)$  (where the upper index  $h$  denotes the horizontal lift extended by zero on vertical arguments) ensure that  $(M, A^h, \pi^h)$  is a quasi-classical, generalized CRF structure.

A classical almost contact structure [1] is defined by a triple  $(A \in \text{End}(TM), Z \in \chi(M), \xi \in \Omega^1(M))$  that satisfies (3.2) for  $h = 1, \pi = 0, \sigma = 0$ , i.e.,

$$A^2 = -Id + \xi \otimes Z, AZ = 0, A^* \xi = 0, \xi(Z) = 1.$$

Furthermore, the structure is normal if the normality conditions (3.3) hold, under the same restrictions. Then, we remain with the single condition

$$\mathcal{N}_A(X, Y) + d\xi(X, Y)Z = 0, \forall X, Y \in \chi(M) \quad (3.5)$$

and it implies

$$L_Z A = 0, L_Z \xi = 0, (L_{AX} \xi)(Y) = (L_{AY} \xi)(X).$$

It follows easily that the almost contact structure  $(A, Z, \xi)$  is normal iff (i) the F structure  $A$  is of the CR type and (ii)  $L_Z A = 0$ . Indeed, formula (3.4) with  $h = 1$  shows that the first condition (1.11), which is equivalent to  $A$  being of the CR type (see Remark 2.2), coincides with the normality condition (3.5) for arguments  $X, Y \in P = \text{im } A$  ( $\text{im } A$  is defined by  $\xi = 0$ ). Therefore, normality implies (i) and (ii) and the latter imply the normality conditions for arguments in  $P$ . Then, for  $X \in P$  and  $Y = Z$ , (3.5) becomes  $\mathcal{N}_A(X, Z) = \xi([X, Z])Z$  and it is implied by (i) and (ii) because  $L_Z A = 0$  implies the vanishing of both sides of the equality:

$$\begin{aligned}\mathcal{N}_A(X, Z) &= -A[AX, Z] + A^2[X, Z] = A(L_Z A)(X) = 0, \\ \xi([X, Z]) &= \xi([AU, Z]) = -\xi((L_Z A)(U) + A[Z, U]) = 0 \quad (U \in P).\end{aligned}$$

Furthermore, if  $(A, Z, \xi)$  is normal,  $A$  is a classical CRF structure since it is of the CR type and  $\mathcal{N}_A(X, Z) = 0$  for  $X \in P$ .

**Definition 3.1.** Let  $(A, Z, \xi)$  be an almost contact structure on a manifold  $M$ . A Poisson bivector field  $\pi \in \chi^2(M)$  is a *contact-Poisson structure* on  $M$  if  $(A, \pi)$  is a quasi-classical, generalized CRF structure.

The integrability conditions of the quasi-classical structure defined by  $(A, \pi)$  show that the bivector field  $\pi$  is a contact-Poisson structure on  $(M, A, Z, \xi)$  iff: (1)  $\pi(\alpha \circ A, \beta) = \pi(\alpha, \beta \circ A)$ ,  $i(\xi)\pi = 0$ , (2)  $A$  is a classical CRF structure, (3)  $\pi$  is a Poisson bivector field, (4)  $R_{(\pi, A)}(X, \beta) = 0$  whenever  $\xi(X) = 0, \beta(Z) = 0$ . Condition (4) may be replaced by the conditions (2.19). In particular, (4) implies  $\xi([\sharp_\pi \alpha, X]) = 0$  whenever  $\xi(X) = 0$ ; for  $\alpha(Z) = 0$  this is the first condition (2.19) and for  $\alpha = \xi$  we have  $\sharp_\pi \alpha = 0$ .

Of course,  $Q = \ker A = \text{span}\{Z\}$  is a foliation. If we also assume that  $L_Z \pi = 0$ , it easily follows that  $\pi$  is  $Q$ -projectable and we may apply Proposition 3.1 and get the  $Q$  transversal holomorphic Poisson structure defined by the projections of  $A$  and  $\pi$ . This situation occurs for contact-Poisson structures on normal, almost contact manifolds.

**Definition 3.2.** Let  $(A, Z, \xi)$  be a normal almost contact structure on a manifold  $M$  and  $\pi$  a contact-Poisson structure on  $M$ . If  $L_Z \pi = 0$ ,  $\pi$  will be called a *normal contact-Poisson structure*.

**Example 3.1.** Let  $(M, A, Z, \xi)$  be a normal almost contact manifold such that  $\xi \wedge d\xi = 0$ . Then,  $P = \text{im } A$  is integrable and, since  $A$  is of the CR type,  $A|_P$  defines complex structures on the leaves of  $P$ . Any  $Z$ -projectable bivector field  $\pi$  that produces holomorphic Poisson structures of the leaves of  $P$  obviously is a normal contact-Poisson structure on  $M$ . In particular, this is the case for a cosymplectic manifold in the sense of Blair [1] because such a manifold satisfies the condition  $d\xi = 0$ . The structure  $\pi = 0$  is a trivial example of a normal contact-Poisson structure on any almost contact manifold  $M$ , where the subbundle  $P$  may not be involutive. This trivial example can be modified as follows. Take  $M = N \times N'$  where  $N$  is a complex manifold with the complex structure tensor  $J$  and the holomorphic Poisson structure  $\pi \neq 0$  and  $N'$  is a normal almost contact manifold with the structure  $(A, Z, \xi)$ , where  $\text{im } A$  may not be involutive. Then,  $(\tilde{A} = J + A, Z, \xi)$  is still a normal almost contact manifold and  $\text{im } \tilde{A}$  may not be involutive. It is easy to check that  $\pi$  is a normal contact-Poisson structure on  $M$ .

**Proposition 3.3.** If  $(M, A, Z, \xi)$  is a normal almost contact manifold and  $\pi$  is a normal contact-Poisson structure on  $M$ , then, the generalized almost contact structure  $(A, \pi, Z, \xi)$  is normal.

*Proof.* Under the hypotheses, we already have conditions (3.3) except for

$$L_{\sharp_\pi \alpha} \xi = 0, R_{(\pi, A)}(X, \beta) = 0, \forall (X, \alpha). \quad (3.6)$$

The first condition (3.6) follows since, for  $\xi(X) = 0$ , an already mentioned consequence of property (4) of contact-Poisson structures gives

$$\langle L_{\sharp_\pi \alpha} \xi, X \rangle = - \langle \xi, [\sharp_\pi \alpha, X] \rangle = 0 \quad (3.7)$$

and

$$\langle L_{\sharp\pi\alpha}\xi, Z \rangle = -\langle \xi, [\sharp\pi\alpha, Z] \rangle = \langle \xi, \sharp_{L_Z\pi}\alpha + \sharp\pi(L_Z\alpha) \rangle = -\langle L_Z\alpha, \sharp\pi\xi \rangle = 0.$$

Furthermore, the second condition (3.6) holds under the restrictions of (4) and we have to check it for the arguments that do not satisfy these restrictions:  $(A^2X, \xi), (Z, A^{*2}\alpha), (Z, \xi)$ . We have

$$R_{(\pi, A)}(A^2X, \xi) = \sharp\pi[L_{A^2X}(A^*\xi) - L_{A^3X}\xi] - (L_{\sharp\pi\xi}A)(A^2X) = \sharp\pi(L_{AX}\xi)$$

and, also, for all  $\alpha$ ,

$$\langle \alpha, \sharp\pi(L_{AX}\xi) \rangle = -\langle L_{AX}\xi, \sharp\pi\alpha \rangle = \langle \xi, [\sharp\pi\alpha, AX] \rangle = 0.$$

Then,

$$\begin{aligned} R_{(\pi, A)}(Z, A^{*2}\alpha) &= \sharp\pi[L_Z(A^{*3}\alpha) - L_{AZ}\alpha] - (L_{\sharp\pi(A^{*2}\alpha)}A)(Z) \\ &= -\sharp\pi(A^*L_Z\alpha) - [\sharp\pi(A^{*2}\alpha), AZ] + A[\sharp\pi(A^{*2}\alpha), Z] \\ &= -A\sharp\pi(L_Z\alpha) - A(\sharp_{L_Z\pi}(A^{*2}\alpha) - A^3\sharp\pi(L_Z\alpha)) = 0. \end{aligned}$$

Finally,  $R_{(\pi, A)}(Z, \xi) = 0$  follows straightforwardly from the definition of  $R_{(\pi, A)}$ .  $\square$

If  $(M_u, A_u, Z_u, \xi_u)$ ,  $u = 1, 2$ , are two almost contact manifolds, the formula

$$J(X_1, X_2) = (A_1X_1 - \xi_2(X_2)Z_1, A_2X_2 + \xi_1(X_1)Z_2),$$

where  $X_u \in T_{x_u}M_u$ ,  $x_u \in M_u$ ,  $u = 1, 2$ , defines an almost complex structure on  $M_1 \times M_2$  and it was proven in [10] that  $J$  is integrable iff the two almost contact structures are normal. In this case, the following proposition is true.

**Proposition 3.4.** *Let  $(M_u, A_u, Z_u, \xi_u)$ ,  $u = 1, 2$ , be two normal almost contact manifolds that have normal contact-Poisson structures  $\pi_u$ . Then,  $\pi = \pi_1 + \pi_2$  is a holomorphic Poisson structure on the complex analytic manifold  $(M_1 \times M_2, J)$ .*

*Proof.* Obviously,  $\pi$  is a Poisson structure. Hence, conditions (1.5) reduce to

$$R_{(\pi, J)}((X_1, X_2), (\alpha_1, \alpha_2)) = 0, \quad (3.8)$$

where, because  $R$  is a tensor on  $M_1 \times M_2$ , we may assume that  $X_u \in \chi(M_u)$ ,  $\alpha_u \in \Omega^1(M_u)$ . We need to check (3.8) for the following type of arguments: (1)  $(X_1, 0), (\alpha_1, 0)$ , (2)  $(X_1, 0), (0, \alpha_2)$ , (3)  $(0, X_2), (\alpha_1, 0)$  (4)  $(0, X_2), (0, \alpha_2)$ .

Firstly, we notice the expression of the transposed operator  $J^*$ :

$$J^*(\alpha_1, \alpha_2) = (A_1^*\alpha_1 + \alpha_2(Z_2)\xi_1, A_2^*\alpha_2 - \alpha_1(Z_1)\xi_2).$$

Now, using the definition (1.7) of the Schouten concomitant, a straightforward calculation gives

$$R_{(\pi, J)}((X_1, 0), (\alpha_1, 0)) = (R_{(\pi_1, A_1)}(X_1, \alpha_1), \xi_1([\sharp\pi_1\alpha_1, X_1])Z_2),$$

which vanishes because the normality of the first manifold implies  $R_{(\pi_1, A_1)}(X_1, \alpha_1) = 0$  and (3.7) for the first manifold is  $\xi_1([\sharp_{\pi_1} \alpha_1, X_1]) = 0$ . In the same way, we will obtain (3.8) in case (4).

Then, starting with (1.7) we get

$$\begin{aligned} R_{(\pi, J)}(X_1, 0), (0, \alpha_2) &= (\alpha_2(Z_2) \sharp_{\pi_1}(L_{X_1} \xi_1), 0) - (0, \xi_1(X_1) \sharp_{\pi_2}(L_{Z_2} \alpha_2)) \\ &\quad - (0, \xi_1(X_1) [\sharp_{\pi_2} \alpha_2, Z_2]). \end{aligned}$$

The property  $L_{Z_2} \pi_2 = 0$  implies the cancelation of the last two terms and  $\sharp_{\pi_1}(L_{X_1} \xi_1) = 0$  follows from the evaluation on any 1-form  $\beta_1$ :

$$\langle \sharp_{\pi_1}(L_{X_1} \xi_1), \beta_1 \rangle = - \langle L_{X_1} \xi_1, \sharp_{\pi_1} \beta_1 \rangle = \langle \xi_1, [X_1, \sharp_{\pi_1} \beta_1] \rangle \stackrel{(3.7)}{=} 0.$$

Thus, (3.8) holds in case (2). In case (3), (3.8) follows similarly.  $\square$

## 4 Poisson cohomology

In this section we make some remarks on the Poisson cohomology defined by the Poisson structure  $\pi$  of a quasi-classical generalized CRF manifold  $(M, A, \pi)$ . This is a particular case of a more general situation worthy of consideration.

In an older terminology, a subbundle  $P \subseteq TM$  is a (non)holonomic submanifold of  $M$  (“non” is omitted in the involutive case).

**Definition 4.1.** A (non)holonomic submanifold  $P$  of the Poisson manifold  $(M, \pi)$  is a *(non)holonomic Poisson submanifold* if: 1)  $\text{im}(\sharp_{\pi} : T^*M \rightarrow TM) \subseteq P$ , 2) every infinitesimal transformation  $\sharp_{\pi} \alpha$  ( $\alpha \in \Omega^1(M)$ ) preserves the distribution  $P$ .

Condition 2) means that

$$[\sharp_{\pi} \alpha, X] \in P, \quad \forall \alpha \in \Omega^1(M), \quad X \in P. \quad (4.1)$$

If condition 1) holds, condition 2) is coherent since (4.1) is consistent with the multiplication of either  $\alpha$  or  $X$  by a function.

If  $\Phi$  is a quasi-classical generalized CRF structure defined by the pair  $(A, \pi)$ , Proposition 2.1 and Proposition 2.4 show that  $P = \text{im } A$  is a (non)holonomic Poisson submanifold of  $(M, \pi)$ .

If  $\pi$  of Definition 4.1 is non-degenerate, we have  $P = TM$  and (4.1) automatically holds. If  $\pi$  is a regular Poisson structure and  $P = \text{im } \sharp_{\pi}$ ,  $P$  is the symplectic foliation of  $\pi$  (hence, holonomic) and (4.1) again holds because  $X$  is tangent to the symplectic leaves of  $\pi$ . In the general holonomic case,  $P$  is integrable and condition 1) of Definition 4.1 means that every leaf of  $P$  is a Poisson submanifold, i.e., a union of open subsets of symplectic leaves of  $\pi$  [2]. Condition 2) is again implied because the brackets compute along the leaves of  $P$ . Thus, the notion of a (non)holonomic Poisson submanifold is a natural extension of the notion of a Poisson submanifold.

An example of a nonholonomic Poisson submanifold is given below.



**Example 4.1.** Take  $M = \mathbb{R}^5$  with the cartesian coordinates  $(y^1, y^2, x^1, x^2, x^3)$  and the Poisson bivector field

$$\pi = f(y^1, y^2, x^1, x^2, x^3) \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}.$$

Consider the distribution

$$P = \text{span}\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, X_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, X_2 = \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3}\right\}.$$

Then,  $P$  includes  $\text{im } \sharp_\pi$ , which is zero where  $f$  vanishes and is spanned by  $\partial/\partial y^1, \partial/\partial y^2$  where  $f \neq 0$ .  $P$  satisfies condition 2) of Definition 4.1 since  $[\partial/\partial y^a, X_u] = 0$  for  $a = 1, 2, u = 1, 2$  and  $P$  is nonholonomic since  $[X_1, X_2] = -2(\partial/\partial x^3)$  is not contained in  $P$ .

The Poisson cohomology of a Poisson manifold  $(M, \pi)$  is the de Rham cohomology of the Lie algebroid  $(T^*M, \sharp_\pi, \{, \}_\pi)$ , i.e., the cohomology of the cochain complex  $(\chi^k(M), d_\pi)$ , where the coboundary  $d_\pi$  is given by [13]

$$\begin{aligned} d_\pi w(\lambda_0, \dots, \lambda_k) &= \sum_{h=0}^k (-1)^h (\sharp_\pi \lambda^h)(w(\lambda_0, \dots, \hat{\lambda}_h, \dots, \lambda^k)) \\ &\quad + \sum_{h < s} (-1)^{h+s} w(\{\lambda_h, \lambda_s\}_\pi, \lambda_0, \dots, \hat{\lambda}_h, \dots, \hat{\lambda}_s, \dots, \lambda_k), \end{aligned} \quad (4.2)$$

the hat denoting a missing argument. Another expression of the coboundary is  $d_\pi w = -[\pi, w]$  (Schouten-Nijenhuis bracket). In fact, every Lie algebroid  $\mathcal{A}$  has de Rham cohomology spaces  $H^k(\mathcal{A})$  defined by the cochain complex  $(\Gamma \wedge^k \mathcal{A}^*, d_{\mathcal{A}})$ , where  $d_{\mathcal{A}}$  has the same expression as  $d_\pi$ , but,  $\sharp_\pi$  is replaced by the anchor of  $\mathcal{A}$  and the  $\pi$ -bracket is replaced by the bracket of  $\mathcal{A}$ . The Poisson cohomology spaces are  $H^k(M, \pi) = H^k(T^*M)$ .

Concerning the Lie algebroid  $T^*M$ , we notice the following result.

**Proposition 4.1.** *If  $P$  is a (non)holonomic Poisson submanifold of the Poisson manifold  $(M, \pi)$ , the Lie algebroid  $(T^*M, \sharp_\pi, \{, \}_\pi)$  induces a Lie algebroid structure on  $P^* = T^*M/(\text{ann } \pi)$ . Furthermore, if we put anchor and bracket zero on  $\text{ann } P$ , the sequence*

$$0 \rightarrow \text{ann } P \xrightarrow{\iota} T^*M \xrightarrow{p} T^*M/(\text{ann } P) \rightarrow 0, \quad (4.3)$$

where  $\iota$  is the inclusion and  $p$  is the natural projection, is an exact sequence of Lie algebroids over  $M$ .

*Proof.* Equality (2.4) and its consequence that  $\text{im}(\sharp_\pi : T^*M \rightarrow TM) \subseteq P$  is equivalent to  $\sharp_\pi(\text{ann } P) = 0$  hold for any pair  $(\pi \in \chi^2(M), P \subseteq TM)$ . Hence, if  $\text{im } \sharp_\pi \subseteq P$ , we get an induced morphism  $\sharp'_\pi : T^*M/(\text{ann } P) \approx P^* \rightarrow P$ .

Then, since  $\sharp_\pi(\text{ann } P) = 0$ , formula (2.9) shows that,  $\forall \alpha, \beta \in \text{ann } P$  one has  $\{\alpha, \beta\}_\pi = 0$ . Moreover, if only one of the forms, say  $\beta$ , belongs to  $\text{ann } P$ , then,  $\forall \gamma \in \Omega^1(M)$ ,

$$\begin{aligned} \langle \gamma, \sharp_\pi \{\alpha, \beta\}_\pi \rangle &= \langle \gamma, \sharp_\pi (L_{\sharp_\pi \alpha} \beta) \rangle = - \langle L_{\sharp_\pi \alpha} \beta, \sharp_\pi \gamma \rangle \\ &= (\sharp_\pi \alpha)(\pi(\beta, \gamma)) + \langle \beta, [\sharp_\pi \alpha, \sharp_\pi \gamma] \rangle = \langle \beta, \sharp_\pi \{\alpha, \gamma\}_\pi \rangle = 0, \end{aligned}$$

where the penultimate equality sign is justified by the fact that  $\pi$  is Poisson and the ultimate is justified by  $\sharp_\pi(\text{ann } P) = 0$ . Thus,  $\{\alpha, \beta\}_\pi \in \text{ann } \sharp_\pi$ .

Accordingly, the formula

$$\{[\alpha], [\beta]\}_\pi = [\{\alpha, \beta\}_\pi],$$

where brackets denote classes in  $T^*M/(\text{ann } P)$ , yields a well defined bracket on  $\Gamma P^*$ . Obviously, if the Jacobi identity holds for the bracket of 1-forms, which is true if  $\pi$  is Poisson, it also holds for the bracket of classes.

The above proves the existence of the induced Lie algebroid structure of  $P^*$ . The exactness of the sequence (4.3) is obvious.  $\square$

**Remark 4.1.** Proposition 4.1 may be generalized as follows. Let  $L \subseteq \mathbf{TM}$  be a Dirac structure (i.e., a maximally  $g$ -isotropic subbundle closed under Courant brackets) on a manifold  $M$  and  $S = \text{pr}_{TM} L$ . For any (non)holonomic submanifold  $P \subseteq TM$ , we denote [14]:

$$\begin{aligned}\tilde{H}(L, P) &= \{(X, \alpha) \in L, / \alpha \in \text{ann } P\} = L \cap (TM \oplus \text{ann } P) \subseteq \mathbf{TM}, \\ H(L, P) &= \{X \in TM, / \exists \alpha \in \text{ann } P, (X, \alpha) \in L\} = \text{pr}_{TM} \tilde{H}(L, P).\end{aligned}$$

Then, we will say that  $P$  is a *(non)holonomic Dirac submanifold* of  $(M, L)$  if (1)  $S \subseteq P$ , (2) for any vector fields  $Z \in S, X \in P, [Z, X] \in P$ . Then, if  $L \cap (\text{ann } P)$  is a differentiable vector bundle ( $\text{ann } P$  is identified with a subspace of  $TM \oplus T^*M$  by  $\alpha \mapsto (0, \alpha)$  ( $\alpha \in \text{ann } P$ )), the sequence

$$0 \rightarrow L \cap (\text{ann } P) \xrightarrow{\iota} L \xrightarrow{p} L/L \cap (\text{ann } P) \rightarrow 0, \quad (4.4)$$

with the brackets induced by the Courant bracket is an exact sequence of Lie algebroids. By (1.2), the restriction of the Courant bracket to  $T^*M$ , and in particular to pairs  $(0, \alpha), (0, \beta) \in L \cap \text{ann } P$ , is zero. Also, for  $\alpha \in \text{ann } P$  and  $(Y, \beta) \in L$ , we get

$$[(0, \alpha), (Y, \beta)] = (0, -L_Y \alpha) \in L \cap \text{ann } P$$

because, if  $Z \in P$ ,

$$L_Y \alpha(Z) = - \langle \alpha, [Y, Z] \rangle \stackrel{(2)}{=} 0.$$

Thus, the Courant bracket induces a Lie algebroid bracket on the quotient that appears in (4.4). The case of a Poisson manifold  $(M, \pi)$  is  $L = \text{graph } \sharp_\pi$ .

In the proof of Proposition 4.1 we saw that, if  $P$  is a (non)holonomic Poisson submanifold of  $(M, \pi)$ ,  $\text{ann } P$  is an abelian ideal of  $(\Omega^1(M), \{\cdot, \cdot\}_\pi)$ . This property was encountered in the case  $P = \text{im } \sharp_\pi$ , where  $\pi$  is a regular Poisson structure, and it was used to define a spectral sequence that converges to the Poisson cohomology of  $\pi$  [13]. Below, we recall the definition of this spectral sequence, referring to an arbitrary (non)holonomic Poisson submanifold. As a matter of fact, the construction of the spectral sequence involves a complementary subbundle  $Q$  of  $P$ , which yields the decomposition  $TM = Q \oplus P$ . In

the case of a quasi-classical generalized CRF structure with associated tensors  $(A, \pi)$ , one has a canonical spectral sequence defined by  $P = \text{im } A, Q = \text{ker } A$ . The decomposition  $TM = P \oplus Q$  implies

$$T^*M = Q^* \oplus P^* = (\text{ann } P) \oplus (\text{ann } Q).$$

Accordingly, we may transfer the Lie algebroid structure of  $P^*$  given by Proposition 4.1 to  $\text{ann } Q$  and replace  $L/L \cap (\text{ann } P)$  by  $\text{ann } Q$  in (4.4).

The decomposition of  $T^*M$  produces a bi-grading  $\chi^k(M) = \sum_{i+j=k} \chi^{ij}(M)$ , where  $i, j \geq 0$  and  $i$  is the  $Q$ -degree,  $j$  is the  $P$ -degree (i.e.,  $w \in \chi^{ij}(M)$  vanishes unless evaluated on  $i$  1-forms in  $Q^* = \text{ann } P$  and  $j$  1-forms in  $P^* = \text{ann } Q$ ). Then, since  $\sharp_\pi(\text{ann } P) = 0$  and  $\text{ann } P$  is an abelian ideal of  $(\Omega^1(M), \{, \}_\pi)$ , if we count the arguments in  $d_\pi w$  given by (4.2) for  $w \in \chi^{ij}(M)$ , we see that  $d_\pi = \sigma'_{-1,2} + \sigma''_{0,1}$ , where the lower indices indicate the grade increments. Furthermore, the coboundary property  $d_\pi^2 = 0$  is equivalent to

$$\sigma'^2 = 0, \sigma''^2 = 0, \sigma' \circ \sigma'' + \sigma'' \circ \sigma' = 0. \quad (4.5)$$

Formula (4.2) yields the following expression of  $\sigma', \sigma''$  for  $w \in \chi^{ij}(M)$  [13]

$$\begin{aligned} & (\sigma' w)(\alpha_0, \dots, \alpha_{i-2}, \beta_0, \dots, \beta_{j+1}) \\ &= \sum_{h < k=0}^{j+1} (-1)^{h+k} w(\{\beta_h, \beta_k\}_\pi, \alpha_0, \dots, \alpha_{i-2}, \beta_0, \dots, \hat{\beta}_h, \dots, \hat{\beta}_k, \dots, \beta_{j+1}), \\ & (\sigma'' w)(\alpha_0, \dots, \alpha_{i-1}, \beta_0, \dots, \beta_j) \\ &= \sum_{h=0}^j (-1)^{i+h} (\sharp_\pi \beta_h)(w(\alpha_0, \dots, \alpha_{i-1}, \beta_0, \dots, \hat{\beta}_h, \dots, \beta_j) \\ &+ \sum_{h=0}^{i-1} \sum_{k=0}^j (-1)^{i+h+k} w(\{\alpha_h, \beta_k\}_\pi, \alpha_0, \dots, \dots, \hat{\alpha}_h, \dots, \alpha_{i-1}, \beta_0, \dots, \hat{\beta}_k, \dots, \beta_j) \\ &+ \sum_{h < k=0}^j (-1)^{h+k} w(\alpha_0, \dots, \alpha_{i-1}, \{\beta_h, \beta_k\}_\pi, \beta_0, \dots, \hat{\beta}_h, \dots, \hat{\beta}_k, \dots, \beta_j), \end{aligned} \quad (4.6)$$

where the arguments  $\alpha \in \text{ann } P$  and  $\beta \in \text{ann } Q$ .

**Remark 4.2.** The restriction of  $\sigma''$  to  $\sum_j \chi^{0j}(M)$  defines the cohomology of the Lie algebroid  $P^*$ . On the other hand,  $H^j(\text{ann } P) = \Gamma \wedge^j Q$  since  $\text{ann } P$  has zero anchor and bracket.

Now, we define a filtration of the cochain complex  $(\chi^\bullet(M), d_\pi)$  by the spaces of filtration degree  $h$ :

$$F_h(M) = (\oplus_{i=0}^q \chi^{i,h}) \oplus (\oplus_{i=0}^q \chi^{i,h+1}) \oplus \dots \oplus (\oplus_{i=0}^q \chi^{i,p}),$$

where  $q = \text{rank } Q, p = \text{rank } P$ . Obviously,  $F_h(M) \supseteq F_{h+1}(M)$  and the type increments of  $\sigma', \sigma''$  show that  $d_\pi(F_h) \subseteq F_h$ .

The filtration  $F_h(M)$  produces a spectral sequence  $\{E_r^{ij}(M, \pi)\}$ , which converges to the Poisson cohomology of  $(M, \pi)$ . The first terms of the spectral sequence are given by Theorem 5.12 of [13] since the proof of that theorem holds in the present case too:

$$\begin{aligned} E_0^{ij}(M, \pi) &= E_1^{ij}(M, \pi) = \chi^{ij}(M), \\ E_2^{ij}(M, \pi) &= H^i(\chi^{j\bullet}(M), \sigma''). \end{aligned}$$

Notice that  $\chi^{jk}(M) = \Omega^k(P^*, \wedge^j Q)$ , the space of the  $k$ -forms on  $P^*$  with values in  $\wedge^j Q$ .

Furthermore, (4.5) show that  $\sigma'$  induces an action on the cohomology spaces that give  $E_2^{ij}$  and the definitions related with spectral sequences yield

$$E_3^{ij}(M, \pi) = H^i(H^\bullet(\chi^{j\bullet}(M), \sigma''), \sigma').$$

In the case of a quasi-classical generalized CRF manifold  $(M, A, \pi)$  there exists a further grade decomposition induced by  $P = H \oplus \bar{H}$ . In order to express it in a simple way, we consider the local basis  $(h_i \in H, \bar{h}_i \in \bar{H}, q_j \in Q)$  introduced in Proposition 2.2 and the corresponding dual cobasis  $(\kappa^i \in H^*, \bar{\kappa}^i \in \bar{H}^*, \xi^j \in Q^*)$ ;  $(\kappa^i \in H^*, \bar{\kappa}^i \in \bar{H}^*)$  is a basis of  $\text{ann } Q$  and  $(\xi^j)$  is a basis of  $\text{ann } P$ . We will say that  $w \in \chi^k(M)$  is of triple grade  $(a, b, c)$  if its local expression is

$$w = \frac{1}{a!b!c!} w^{l_1 \dots l_a i_1 \dots i_b j_1 \dots j_c} q_{l_1} \wedge \dots \wedge q_{l_a} \wedge h_{i_1} \wedge \dots \wedge h_{i_b} \wedge \bar{h}_{j_1} \wedge \dots \wedge \bar{h}_{j_c},$$

where the coefficients are skew-symmetric in each of the three groups of indices and  $a + b + c = k$ . The space of such multivectors will be denoted by  $\chi^{abc}(M)$ .

**Proposition 4.2.** *For a quasi-classical generalized CRF manifold, one has a decomposition*

$$\sigma'' = \sigma''_H + \sigma''_{\bar{H}}, \quad (4.7)$$

where  $\sigma''_H : \chi^{abc}(M) \rightarrow \chi^{a,b+1,c}(M)$ ,  $\sigma''_{\bar{H}} : \chi^{abc}(M) \rightarrow \chi^{a,b,c+1}(M)$  and  $\sigma''_H{}^2 = \sigma''_{\bar{H}}{}^2 = 0$ ,  $\sigma''_H \circ \sigma''_{\bar{H}} + \sigma''_{\bar{H}} \circ \sigma''_H = 0$ .

*Proof.* We shall use the second formula (4.6) in order to compute  $\sigma''$ , which has type  $(0, 1)$ , for corresponding arguments as follows:

$$\begin{aligned} (\sigma'' w)(\xi^{l_1}, \dots, \xi^{l_a}, \kappa^{i_1}, \dots, \kappa^{i_{b+e}}, \bar{\kappa}^{j_1}, \dots, \bar{\kappa}^{j_{c-e+1}}), \quad e = 1, \dots, c+1, \\ (\sigma'' w)(\xi^{l_1}, \dots, \xi^{l_a}, \kappa^{i_1}, \dots, \kappa^{i_{b-f+1}}, \bar{\kappa}^{j_0}, \dots, \bar{\kappa}^{j_{c+f}}), \quad f = 1, \dots, b+1. \end{aligned}$$

Formula (4.6) straightforwardly shows that the resulting value is zero if either  $e > 2$  or  $f > 2$ . We shall prove the same for  $e = 2, f = 2$ ; the two cases are similar and we give the details for  $e = 2$  only. Then, the first term of the right hand side of the expression (4.6) of  $\sigma''$  vanishes since the involved value of  $w$  does not have the right number of arguments. The second term vanishes because  $\{\xi^l, \kappa^i\}_\pi, \{\xi^l, \bar{\kappa}^i\}_\pi \in \text{ann } P$  imply the addition of one more argument in  $\text{ann } P$  and, again, the number of arguments is not the one that yields non-zero values of  $w$ . For the same reason, we have to replace  $\{\beta_i, \beta_j\}_\pi$  by  $\text{pr}_{\text{ann } Q}\{\beta_i, \beta_j\}_\pi$  in the last term of the same expression of  $\sigma''$ .

In our case,  $\text{ann } Q = H^* + \bar{H}^*$  and we shall get (4.7) and all the required conclusions if we show that  $\text{pr}_{H^*} \text{pr}_{\text{ann } Q}\{\kappa_i, \kappa_j\}_\pi = 0$ . This condition is easily seen to be equivalent to the vanishing of  $\{\kappa_i, \kappa_j\}_\pi(U) \forall U \in \bar{H}$ . We have

$$\begin{aligned} \{\kappa_i, \kappa_j\}_\pi(U) &= \langle L_{\#_\pi \kappa_i} \kappa_j, U \rangle - \langle L_{\#_\pi \kappa_j} \kappa_i, U \rangle - U(\pi(\kappa_i, \kappa_j)) \\ &= -\langle \kappa_j, [\#_\pi \kappa_i, U] \rangle + \langle \kappa_j, [\#_\pi \kappa_i, U] \rangle. \end{aligned}$$

Since a straightforward calculation leads to the relation

$$< L_{\sharp_{\pi}\kappa_i}\kappa_j, U > + < L_U\kappa_j, \sharp_{\pi}\kappa_i > = U(\pi(\kappa_i, \kappa_j)),$$

we get

$$\begin{aligned} \{\kappa_i, \kappa_j\}_{\pi}(U) &= - < L_U\kappa_j, \sharp_{\pi}\kappa_i > - < L_U\kappa_i, \sharp_{\pi}\kappa_j > - U(\pi(\kappa_i, \kappa_j)) \\ &= (L_U\pi)(\kappa_i, \kappa_j) = 0, \end{aligned}$$

because of the second integrability condition (2.19).  $\square$

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